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# The dynamic effect of flux ropes on Rayleigh–Bénard convection

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The interaction between magnetic fields and convection in a fluid heated from below is investigated in an axisymmetric cylindrical geometry. When $R_m$, the magnetic Reynolds number, is large the field is concentrated into a thin rope on the axis of the cylinder. For weak magnetic fields a larger Rayleigh number is necessary to produce a flux rope than that needed for infinitesimal convection. For larger total fluxes, however, the opposite is true and the system is subcritically unstable to steady motions. The results are contrasted with those found by Busse (1975) for the corresponding two-dimensional roll problem.

## 1. Introduction

The theory of the interaction between magnetic fields and thermal convection is of great importance in the study of the solar convective zone. Recent observations (Stenflo 1976; Harvey 1977) have shown that the photospheric granulation and supergranulation are threaded by intermittent but intense flux concentrations that are formed by, and react upon, the convective motions (for a discussion see Galloway, Proctor & Weiss 1977); the precise nature of the dynamic balance that is attained is the main goal of inquiry.

There have been numerous studies of idealized model problems. Thompson (1951), Chandrasekhar (1961) and Danielson (1961) have studied the linear instability of a Boussinesq fluid layer heated uniformly from below (the Rayleigh–Bénard or RB problem) with an imposed vertical magnetic field. They found that if $p_3 \equiv \kappa/\eta < 1$, where $\kappa$ is the thermal conductivity and $\eta$ the magnetic diffusivity, motion occurs first as steady convection as the Rayleigh number (a dimensionless measure of the temperature difference across the layer) is increased. If $p_3 > 1$, however, and there is sufficient magnetic flux present, then convection occurs first as overstable oscillations. All these linear results are independent of the convection planform.

More recent investigations of the nonlinear regime have shown that the linear results are rather unrepresentative. When $p_3 \gg 1$ the effects of advection of the magnetic field quickly come to dominate those of diffusion (i.e. the magnetic Reynolds number $R_m \equiv UL/\eta \gg 1$, where $L$ and $U$ are typical length and velocity scales); then

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the flux is expelled from recirculating eddies and becomes confined to thin ropes or sheets between them (Weiss 1966). The geometry of the convection planform now becomes of crucial importance since ropes and sheets have quite different dynamical properties (Galloway, Proctor & Weiss 1978) and ropes, in particular, cannot be formed by a two-dimensional roll pattern. An axisymmetric geometry can be used as a model for a hexagonal planform (Jones, Moore & Weiss 1976). Galloway *et al.* (1978) show by considering the simple Oberbeck problem (in which a steady temperature distribution is prescribed throughout the cell) that axisymmetric ropes affect the local velocity field more strongly for a given flux, and that their back-reaction on the velocity field is largely confined to the flux rope itself.

Galloway (1977) and Galloway & Moore (1979) have described numerical experiments for the full RB problem in an axisymmetric geometry. Among many results, they find that even for large $p_3$ a steady solution is possible with the flux confined to a rope on the axis, even though linear theory predicts overstability, provided the convection is sufficiently vigorous. Very similar phenomena occur for the related problem of thermohaline convection, recently treated in great detail by Huppert & Moore (1976).

However, calculations for the analogous two-dimensional RB problem (Weiss 1975) show that oscillatory motion may persist well into the nonlinear regime. Analytical results are very elusive, but Busse (1975) has investigated the two-dimensional RB problem for $p_3 \gg 1$ and very small magnetic fluxes. The whole solution except the magnetic lines of force can then be calculated analytically when $UL/\kappa$, the Péclet number, is small. Busse finds that when $R_m \gg 1$ convection is possible at smaller values of the Rayleigh number than for the linear problem (for which $R_m$ is perforce vanishingly small), and conjectures that this shows steady motions to be preferred to overstable oscillations at finite amplitude. No definite statement can be made, unfortunately, since the flux is so small in his model that overstability is not possible in the linear regime.

The purpose of the present paper is to attempt to make further progress along the lines advocated by Busse but using the more relevant cylindrical geometry. The motive is to obtain as much information as possible about the evolved steady solutions, even though these may be unstable to unsteady modes for some parameter values, so that when the much harder oscillatory problem is eventually tackled there will be some basis for comparison. Our analysis is formally valid over a much wider parameter range than that of Busse, since we can take over the analytical results used by Galloway *et al.* (1978) for the Oberbeck problem. The analysis becomes tractable owing to the localized nature of the Lorentz forces due to the flux rope. Our results differ qualitatively from those of Busse, indicating the importance of geometry in the flux-rope regime. In particular, for the axisymmetric case higher values of $R_m$ can be obtained only for larger temperature contrasts, when magnetic fluxes are small. For larger fluxes, though, subcritical bifurcation is possible, which is consistent with the existence of finite amplitude steady states. Unfortunately our analysis cannot be extended quite far enough to allow a conclusive demonstration.

There are other useful aspects of the analysis. It complements the work of Galloway (1977) and Galloway & Moore (1979) since it is most accurate precisely where their numerical techniques fail (our basic assumption being that the flux rope is vanishingly thin). It also leads to results for the peak magnetic field in the system as a function of the mean flux, and to an expression for the maximum of this function (criteria of this kind may be useful in predicting the largest fields that can be observed in the solar photosphere; see Galloway *et al.* 1977). Finally, we are hopeful that the techniques can be extended to the much more difficult (but more relevant) compressible convection problem.

The plan of the paper is as follows. Section 2 contains a description of the problem and of the approximations needed to make analytical progress. In § 3 the solution is obtained to the required order using boundary-layer theory and matched asymptotic expansions. In § 4 numerical results for a given (large) value of $p_3$ are presented and discussed. In a conclusion we relate our results to those of Busse and discuss further avenues of exploration.

## 2. Formulation of the problem

We consider convection with an axisymmetric, poloidal velocity field $\mathbf{U}$ and magnetic field $\mathbf{B}$ in a Boussinesq fluid of temperature $T$ and density $\rho = \rho_0\,[1-\alpha(T-T_0)]$, where $T_0$ and $\rho_0$ are constants. The equations satisfied by these quantities are then (cf. Galloway *et al.* 1978)

$$\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial(T,\psi)}{\partial(r,z)} + \kappa\nabla^2 T, \tag{2.1}$$

$$\frac{\partial \chi}{\partial t} = \frac{1}{r}\frac{\partial(\chi,\psi)}{\partial(r,z)} + \eta D^2\chi, \tag{2.2}$$

$$\frac{\partial \omega}{\partial t} = \frac{\partial(\omega/r,\psi)}{\partial(r,z)} - \frac{1}{\rho_0}\frac{\partial(j/r,\chi)}{\partial(r,z)} - g\alpha\frac{\partial T}{\partial r} + \frac{\nu}{r}D^2(r\omega), \tag{2.3}$$

where $\kappa$ and $\eta$ are the thermal and magnetic diffusivities, $\nu$ is the kinematic viscosity and $(r, \phi, z)$ are cylindrical polar co-ordinates. The gravitational acceleration $\mathbf{g} = -g\hat{\mathbf{z}}$ and the Stokes stream function $\psi$, Stokes flux function $\chi$, vorticity $\omega$ and electric current $j$ are defined by

$$\left.\begin{aligned} \mathbf{U} &= \nabla\wedge(\psi r^{-1}\hat{\boldsymbol{\phi}}), \quad \mathbf{B} = \nabla\wedge(\chi r^{-1}\hat{\boldsymbol{\phi}}), \\ \omega &= (\nabla\wedge\mathbf{U}).\hat{\boldsymbol{\phi}} = -r^{-1}D^2\psi, \quad j = (\nabla\wedge\mathbf{B}).\hat{\boldsymbol{\phi}} = -r^{-1}D^2\chi, \end{aligned}\right\} \tag{2.4}$$

where
$$D^2 \equiv r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}$$
is the Stokes operator.

The flow takes place in the cylinder $\{0 \lesssim r \lesssim r_0; 0 \lesssim z \lesssim d\}$. The boundary conditions are chosen to simplify the analysis, and to fix attention on the magnetic flux threading the cylinder. Normal velocity and tangential stress vanish on the boundaries; the curved surface is a perfect electrical conductor; hence the total vertical flux is fixed and equal to that for a uniform field $B_0\hat{\mathbf{z}}$. The field lines are constrained to be vertical at $z = 0, d$, though they emerge at any value of $r$. This convenient condition seems less arbitrary than any procedure that anchors the flux lines. It also implies vanishing Maxwell stresses at the boundary, and so avoids any discussion of the influence of the field on the material outside. Thus

$$\left.\begin{aligned} &\psi = \omega = 0 \quad (z = 0, d;\ r = 0, r_0), \\ &\chi = 0 \quad (r = 0), \quad \chi = \tfrac{1}{2}B_0 r_0^2 \quad (r = r_0), \quad \partial\chi/\partial z = 0 \quad (z = 0, d). \end{aligned}\right\} \tag{2.5}$$

The temperature is prescribed at the upper and lower boundaries and there is no heat flux across the curved surface; thus

$$T = T_0 \quad (z = d), \quad T = T_0 + \Delta T \quad (z = 0), \quad \partial T/\partial r = 0 \quad (r = 0, r_0). \tag{2.6}$$

It is clear that the problem as posed is highly nonlinear, and we need to make some restrictions in the parameter space in order to achieve analytical progress. Our first simplification is to suppose that the Prandtl number $p_1 = \nu/\kappa$ is so large that the Reynolds number

$$Re = UL/\nu, \tag{2.7}$$

where $U$ and $L$ are typical scales for velocity and length, is always vanishingly small. It may be verified that the effect on (2.3) is that both $\partial\omega/\partial t$ and the Jacobian involving $\omega$ on the right-hand side may be ignored in comparison with the other terms, leading to the reduced equation

$$\frac{1}{r}D^2(r\omega) = \frac{g\alpha}{\nu}\frac{\partial T}{\partial r} + \frac{1}{\nu\rho_0}\frac{\partial(j/r, \chi)}{\partial(r, z)} \tag{2.8}$$

in place of (2.3). We also suppose that the motions are sufficiently weak that the deviation of the temperature profile from $T_0 + \Delta T(1 - z/d)$ (its value when $\mathbf{U} = 0$) is very small; i.e. the Péclet number

$$\epsilon = UL/\kappa \tag{2.8}$$

is supposed to be much less than unity. Finally, the Ohmic diffusivity $\eta$ is supposed to be so small that although the Péclet number is small the magnetic field *is* significantly changed from $B_0\hat{\mathbf{z}}$ (its value when $\mathbf{U} = 0$), so that the magnetic Reynolds number

$$R_m = \epsilon\kappa/\eta \tag{2.9}$$

is much *greater* than unity. Of course, these restrictions have implications for the relative magnitudes of the diffusivities $\nu$, $\kappa$ and $\eta$; in fact

$$\nu \gg \kappa \gg \eta \tag{2.10}$$

for (2.7)–(2.10) to be valid.

It is helpful to cast the equations in dimensionless form. Writing

$$t = d^2/\kappa t', \quad (r, z) = d(r', z'), \quad T - T_0 = \Delta T(1 - z' + \epsilon\theta'),$$
$$\mathbf{U} = (\epsilon\kappa/d)\,\mathbf{U}', \quad \mathbf{B} = B_0\mathbf{B}', \text{etc.}, \tag{2.11}$$

substituting into (2.1), (2.2) and (2.8), and dropping primes, we have

$$\frac{\epsilon}{r}\frac{\partial(\psi, \theta)}{\partial(r, z)} = \frac{1}{r}\frac{\partial\psi}{\partial r} + \nabla^2\theta - \frac{\partial\theta}{\partial t}, \tag{2.12}$$

$$\frac{R_m}{r}\frac{\partial(\psi, \chi)}{\partial(r, z)} = D^2\chi - p_3\frac{\partial\chi}{\partial t}, \tag{2.13}$$

$$R\frac{\partial\theta}{\partial r} + QR_m^{-1}\frac{\partial(\chi, r^{-2}D^2\chi)}{\partial(r, z)} = \frac{1}{r}D^2(r\omega), \tag{2.14}$$

where the new dimensionless parameters are the Rayleigh number

$$R = g\alpha\Delta T d^3/\kappa\nu, \tag{2.15}$$

the Chandrasekhar number

$$Q = B_0^2 d^2/\mu\rho_0\nu\eta \tag{2.16}$$

and the diffusivity ratio

$$p_3 = \kappa/\eta. \tag{2.17}$$

As previously noted, we shall suppose that $\epsilon \ll 1$ and $R_m \gg 1$ in what follows. $R$ is (necessarily) of order $R_0 = \frac{27}{4}\pi^4$ (the linear eigenvalue) and we allow $Q$ to take almost any value subject to a very weak restriction [(4.4) below]. The boundary conditions on the dimensionless variables are the same as those on the dimensional ones except that $\theta$ vanishes at $z = 0, 1$.

Of course, $\epsilon$ and $R_m$ are not externally given parameters. They depend in a complicated way upon $R$, $Q$ and $p_3$ and it is the object of the analysis of the next section to discover these relationships for steady motions and fields.

## 3. Solution for steady fields

### 3.1. *Formulation*

We now restrict attention to steady fields so that the time derivatives drop out of (2.1)–(2.3). It will emerge that the change in $R$ from the basic linear eigenvalue $R_0$ obtained by setting $\epsilon = Q = 0$ is small for all values of $Q$ for which the analysis is valid. Since the effects of non-zero $\epsilon$ are small too, by supposition, we may to leading order treat these two perturbations separately. Thus in what follows we set $\epsilon = 0$. The corrections due to $\epsilon$ will be treated later.

The analysis of this section follows closely the treatment of the Oberbeck problem (in which a horizontal temperature gradient is imposed at $z = 0$) given by Galloway *et al.* (1978). We therefore only summarize the intermediate steps; the reader seeking further enlightenment or justification is referred to the above-mentioned paper. We first decompose the velocity and temperature fields into 'basic' and 'magnetically induced' parts. If we write

$$R = R_0 + R_1, \quad \psi = \psi_0 + \psi_1, \quad \theta = \theta_0 + \theta_1, \tag{3.1}$$

where $R_0$ is the linear eigenvalue and $(\psi_0, \theta_0)$ the linear eigenfunction of the non-magnetic problem, we have

$$R_0 \frac{\partial \theta_0}{\partial r} = \frac{1}{r} D^2(r\omega_0), \tag{3.2a}$$

$$0 = \frac{1}{r}\frac{\partial \psi_0}{\partial r} + \nabla^2 \theta_0 \tag{3.2b}$$

and

$$R_1 \frac{\partial \theta_0}{\partial r} + R_0 \frac{\partial \theta_1}{\partial r} + R_1 \frac{\partial \theta_1}{\partial r} + R_m^{-1} Q \frac{\partial(\chi, r^{-2}D^2\chi)}{\partial(r,z)} = \frac{1}{r} D^2(r\omega_1), \tag{3.3a}$$

$$0 = \frac{1}{r}\frac{\partial \psi_1}{\partial r} + \nabla^2 \theta_1. \tag{3.3b}$$

Note that no approximation has yet been made. In order to determine the problem fully, we fix the (dimensional) radius of the cell $r_0 = \lambda_c d$, where

$$\lambda_c = (2^{\frac{1}{2}}/\pi) j_1, \quad J_1(j_1) = 0, \tag{3.4}$$

such that $j_1 = 3{\cdot}83171\ldots$. This leads in the axisymmetric case to the smallest possible value of $R_0$, namely $\frac{27}{4}\pi^4 = 657{\cdot}51\ldots$, and to the eigenfunctions (see Jones *et al.* 1976)

$$\psi_0 = rJ_1(kr)\sin \pi z, \quad \theta_0 = \frac{k}{k^2+\pi^2} J_0(kr) \sin \pi z \quad (k = \pi/2^{\frac{1}{2}}). \tag{3.5}$$

The Péclet number is now defined completely by the normalization used for $\psi_0$. In order to find $R_1$ (the main object of this section) we must assume that the term $R_1 \partial\theta_1/\partial r$ can be neglected in (3.3*a*) compared with all other terms. It can be verified *a posteriori* that this is always the case provided that all the other conditions are fulfilled, even if $Q = O(1)$. The two-dimensional problem is tractable only when $Q$ is very small, so here we at once see an important difference between the two geometries.

We first note that since $R_m \gg 1$ the magnetic flux is confined almost entirely in a flux rope of thickness $O(R_m^{-\frac{1}{2}})$ at the axis of the cell. There is another boundary layer at the outer edge of the cell, but it contains negligible flux since the mean field on this surface is unchanged by the motion (see discussion in §3 of Galloway *et al.* 1978). The flux rope spreads out at $z = 1$ into a horizontal layer of the same thickness. The dynamical effect of the latter layer is small but hard to assess exactly. Clearly it is at most comparable to that of the flux sheets of the roll geometry, which Galloway *et al.* have shown to become important when $Q \sim R_m^{\frac{1}{2}}$. This could conceivably prove a more restrictive condition than (4.5) for the validity of the analysis; however, numerical experiments suggest that this layer is far less important in practice, so we ignore it in what follows.

### 3.2. *The flux rope*

Near the axis, since the rope is of thickness $R_m^{-\frac{1}{2}}$ we may define the stretched coordinate

$$\xi = R_m^{\frac{1}{2}} r. \tag{3.6}$$

The structure of the rope depends only on the vertical velocity at the axis; if

$$\left.\begin{aligned} \psi &= \tfrac{1}{2}r^2 f(z) + o(r^2) \quad \text{as} \quad r \to 0, \\ \text{where} \qquad f &= f_0 + f_1, \quad f_0(z) = k \sin \pi z, \end{aligned}\right\} \tag{3.7}$$

then the equation for $\chi$ is

$$\xi \frac{\partial}{\partial \xi}\left(\frac{1}{\xi}\frac{\partial \chi}{\partial \xi}\right) = f\frac{\partial \chi}{\partial z} - \frac{1}{2}\frac{df}{dz}\xi\frac{\partial \chi}{\partial \xi} \tag{3.8}$$

to leading order. For $1 - z \ll R_m^{-\frac{1}{2}}$ this is solved by

$$\left.\begin{aligned} \chi &= \chi_0[1 - \exp(-\tfrac{1}{2}p\xi^2)], \\ \text{where} \qquad \chi_0 &= \tfrac{1}{2}\lambda_c^2 = j_1^2/\pi^2, \quad p(z) = f/2z. \end{aligned}\right\} \tag{3.9}$$

The function $p(z)$ is related to the vertical field $B^*$ on the axis since

$$B^*(z) = B_0 B_z(0, z) = B_0 R_m p(z) \tag{3.10}$$

from (3.9).

This solution can now be used in (3.3*a*) to find the vorticity generated in the flux rope by the Lorentz forces. The terms in $\theta$ may be neglected in this layer, and the solution for the vorticity $\tilde{\omega}(\xi, z)$ is found to be

$$r\tilde{\omega} = R_m^{-\frac{1}{2}}\xi\tilde{\omega} = \tfrac{1}{2}Q\chi_0^2[1 - \exp(-p\xi^2)]\,dp/dz \tag{3.11}$$

to leading order; thus at the edge of the flux rope we have

$$r\tilde{\omega} \to \tfrac{1}{2}Q\chi_0^2\,dp/dz. \tag{3.12}$$

### 3.3. *The field-free region*

The magnetic field is negligible away from the flux rope. Thus in the interior of the fluid we must satisfy the equations

$$R_1 \frac{\partial \theta_0}{\partial r} + R_0 \frac{\partial \hat{\theta}}{\partial r} = \frac{1}{r} D^2(r\hat{\omega}), \tag{3.13a}$$

$$0 = \frac{1}{r}\frac{\partial \hat{\psi}}{\partial r} + \nabla^2 \theta, \tag{3.13b}$$

where $(\hat{\psi}, \hat{\theta})$ are the 'outer' parts of $(\psi_1, \theta_1)$ respectively. In accordance with the principle of matched asymptotic expansions, we require that as $r \to 0$

$$r\hat{\omega} \to \tfrac{1}{2} Q \chi_0^2 \, dp/dz. \tag{3.14}$$

This boundary condition can in fact be used to determine $R_1$. Before doing this, though, we can find the correction to the vertical velocity on the axis caused by the presence of the rope. Since near $r = 0$

$$D^2 \hat{\psi} = -r\hat{\omega} \sim -\tfrac{1}{2} Q \chi_0^2 \, dp/dz, \tag{3.15}$$

it follows that

$$\hat{\psi} = -\tfrac{1}{4} Q \chi_0^2 r^2 \ln r \, dp/dz + O(r^2) \tag{3.16}$$

near the axis. Now the boundary-layer solution for $\tilde{\omega}$ from (3.11) can be integrated to obtain the boundary-layer stream function $\tilde{\psi}$. Thus we find for large $\xi$

$$\tilde{\psi} = -\tfrac{1}{4} Q \chi_0^2 R_m^{-1} \xi^2 \ln \xi \, dp/dz + \tfrac{1}{2} C(z) R_m^{-1} \xi^2, \tag{3.17}$$

where $C(z)$ is an arbitrary function of integration. In terms of $\xi$, (3.16) may be written as

$$\hat{\psi} = -\tfrac{1}{4} Q \chi_0^2 \frac{dp}{dz} [R_m^{-1} \xi^2 \ln \xi - R_m^{-1} \ln R_m^{\frac{1}{2}} \xi^2 + \dots], \tag{3.18}$$

so the principle of matched asymptotic expansions implies that

$$C(z) = \tfrac{1}{2} Q \chi_0^2 \ln R_m^{\frac{1}{2}} \, dp/dz. \tag{3.19}$$

Now, the particular integral in (3.17) leads to velocities of $O(Q)$ near the axis. Thus the dominant contribution to the velocity correction due to the presence of the rope is

$$f_1(z) = \tfrac{1}{2} Q \chi_0^2 \ln R_m^{\frac{1}{2}} \, dp/dz. \tag{3.20}$$

The analysis leading to (3.20) is treated at much greater length in Galloway *et al.* (1978). Since $f = f_0 + f_1 = 2zp$, we have a relation between $p$, $Q$ and $R_m$:

$$\tfrac{1}{2} Q \chi_0^2 \ln R_m^{\frac{1}{2}} \, dp/dz = 2zp - k \sin \pi z \tag{3.21}$$

which is to be solved subject to the condition $p(1) = 0$. The conditions that

$$f(0) = d^2 f(0)/dz^2 = 0,$$

implied by the velocity boundary conditions, are in fact satisfied by all solutions of (3.21). If $R_m$ is known (as in the Oberbeck case), (3.21) determines $p(z)$, and thus the field on the axis, as a function of $Q$. Here it is the Rayleigh number that is given externally, so we need a further relation to close the problem. This is provided by the solvability condition for (3.13).

### 3.4. *Determination of* $R_1$

It is clear that (3.13) is an inhomogeneous version of the linear eigenvalue problem and thus has a solution only if the inhomogeneous terms are orthogonal to the linear eigensolution (since the basic problem is self-adjoint). If we multiply (3.13*a*) by $\psi_0/r$, (3.13*b*) by $\theta_0$ and integrate over the cylinder, we obtain after some algebra

$$\left.\begin{aligned} CR_1 &= -\frac{Q\chi_0^2}{2}\int_0^1 f_0(z)\frac{dp}{dz}dz, \\ \text{where}\qquad C &= \frac{\lambda_c^2 k^2 J_0^2(j_1)}{4(k^2+\pi^2)} = 0\cdot0402\ldots. \end{aligned}\right\} \tag{3.22}$$

Equation (3.22) is a necessary and sufficient condition for the existence of a solution $\hat{\psi}$ whose expansion about $r = 0$ leads to $f_1(z)$ as calculated in § 3.3 above. The significance of the enigmatic expression on the right-hand side of (3.22) can be clarified. Substitution from (3.21) enables it to be rewritten as

$$CR_1 = \frac{1}{2\pi}\left[2\pi\frac{Q\chi_0^2}{2}\int_0^1 p^2(z)\,dz + 2\pi\frac{Q^2\chi_0^4}{4}\ln R_m^{\frac{1}{2}}\int_0^1\left(\frac{dp}{dz}\right)^2 dz\right]. \tag{3.23}$$

The terms on the right-hand side of (3.23) can be related to more familiar quantities by means of a power integral argument. If we take the full equations (2.12)–(2.14) with $\partial/\partial t = 0$, multiply the second by $r^{-2}D^2\chi$, the third by $\psi/r$ and integrate over the cylinder, we obtain the power integral

$$R\int \mathbf{U}.\hat{\mathbf{z}}\theta dV = \int \omega^2 dV + Q\Omega/R_m^2, \tag{3.24}$$

where $\Omega \equiv \int (r^{-1}D\chi)^2\,dV$ is a measure of the Ohmic dissipation. If we now substitute $\psi = \psi_0+\psi_1$, etc., we find to leading order

$$R_0\int \mathbf{U}_0.\hat{\mathbf{z}}\theta_0 dV = \int \omega_0^2 dV \tag{3.25}$$

and

$$R_1\int \mathbf{U}_0.\hat{\mathbf{z}}\,\theta_0\, dV = Q\Omega/R_m^2 + \int \omega_1^2 dV. \tag{3.26}$$

Although $\omega_1$ is small compared with $\omega_0$ in the interior, it becomes large in the boundary layer and in fact the two terms on the right-hand side of (3.23) are identifiable [as may easily be verified by use of (3.9) and (3.11)] with the respective terms on the right-hand side of (3.26). When $Q\ln R_m^{\frac{1}{2}} = O(1)$ the viscous term is of the same order as the Ohmic one and when this product is large the viscous term actually dominates. The correspondence is a powerful check on the validity of the analysis and also demonstrates that the field is certainly not 'weak' even though $R_1/R_0$ is small.

The expansion of $R$ may now be completed by adding the terms in $\epsilon$. Since

$$R_1 = O(Q|dp/dz|)$$

we may assume that $Qdp/dz = O(\epsilon^2)$ so that the Péclet number correction may be obtained by assuming that $Q = 0$. This problem has been solved by Jones *et al.* (1976) and leads when $\nu \gg \kappa$ to a correction $+\,12\cdot56\epsilon^2$ to leading order in $\epsilon$.

Thus the complete expansion for $R$ takes the form

$$\begin{aligned} R &= \tfrac{27}{4}\pi^4 + 12\cdot56\epsilon^2 + \frac{Q\chi_0^2}{2C}\int_0^1 p^2 dz + \frac{Q^2\chi_0^4}{4C}\ln R_m^{\frac{1}{2}}\int_0^1\left(\frac{dp}{dz}\right)^2 dz \\ &= \tfrac{27}{4}\pi^4 + 12\cdot56\epsilon^2 + \frac{Q\chi_0^2}{2C}\int_0^1 p(z)\frac{df_0}{dz}dz. \end{aligned} \tag{3.27}$$

This equation, together with (3.21) and the identity $R_m = \epsilon p_3$, yields a transcendental relation between $\epsilon$ and $R$, $Q$. and $p_3$. Further progress requires numerical methods, except in certain limiting situations. It is evident, though, that considerable simplification has been achieved over the original nonlinear partial differential equations.

In the next section, we present computed solutions of (3.27) and (3.21) for a number of representative cases. We also derive certain analytical results which are available, and compare the results we obtain with Busse's (1975) calculations for the two-dimensional case. It will be seen that there are important qualitative differences between the two geometries.

## 4. Results and discussion

### 4.1. *Weakly nonlinear (low $R_m$) theory*

It is helpful in setting up the framework for what follows to recall some earlier results by Chandrasekhar (1961) and others. It is known that the critical Rayleigh number for the onset of infinitesimal steady motion is

$$R_c = \tfrac{27}{4}\pi^4 + 3\pi^2 Q \tag{4.1}$$

(since the values of $Q$ studied here are not so large that a convective mode with two nested cells is preferred). Furthermore, it is possible when $\epsilon$ is so small that $R_m = \epsilon p_3 \ll 1$ to find the dependence of $R$ on $\epsilon$ and $R_m$ by perturbation methods similar to those used by Jones *et al.* (1976). When $Q$ is so small that Lorentz forces are negligible, we find that

$$R = R_c + 12{\cdot}56\epsilon^2 + 1{\cdot}32\, R_m^2\, Q, \tag{4.2}$$

so that for small enough $Q$ the bifurcation at $R = R_c$ is definitely *supercritical*. Busse's (1975) analysis (carried out for a *square* two-dimensional roll) gives subcritical behaviour, but for cells with an aspect ratio of $1 : 2^{\frac{1}{2}}$ (corresponding to $R_0 = \frac{27}{4}\pi^4$) the coefficient of $QR_m^2$ was found to be positive (personal communication from F. H. Busse). The present results should *not* be interpreted, though, to imply that axisymmetric steady magnetoconvection is always supercritically unstable, since the small-$R_m$ results show more complicated behaviour when $Q$ is not small.

### 4.2. *The kinematic limit ($Q \ln R_m^{\frac{1}{2}} \ll 1$)*

For sufficiently small $Q$ the deviation from the basic flow $\mathbf{U}_0$ is so small, even in the flux rope, that (3.27) becomes

$$R = \tfrac{27}{4}\pi^4 + 12{\cdot}56\epsilon^2 + \frac{Q\chi_0^2}{2C}\int_0^1 p_0^2\, dz, \tag{4.3}$$

where $p_0 = k \sin \pi z/2z$. The integral can be transformed into one that is tabulated, so that we have

$$R = \tfrac{27}{4}\pi^4 + 12{\cdot}56\epsilon^2 + 153Q. \tag{4.4}$$

This confirms the result (4.2) on the supercriticality of the bifurcation, since the expression in (4.4) is definitely greater than $R_c$. It should be remarked that the two-dimensional result is quite different, the term in $Q$ being replaced by one of the form + constant $Q(\epsilon p_3)^{-\frac{1}{2}}$.

### 4.3. *The large-Q (dynamic) limit*

If $Q \ln R_m^{\frac{1}{2}} \gg 1$ but the flux rope is still thin, so that

$$R_m|p| \sim R_m/Q \ln R_m^{\frac{1}{2}} \gg 1 \tag{4.5}$$

(this provides the restriction on $Q$ referred to in § 2), then from (3.21) we have at leading order

$$dp/dz = -2k \sin \pi z/Q\chi_0^2 \ln R_m^{\frac{1}{2}}, \tag{4.6}$$

so to this order the terms in $Q$ in (3.27) reduce to

$$\frac{k^2}{C}\int_0^1 \sin^2 \pi z\, dz \frac{1}{\ln R_m^{\frac{1}{2}}} = \frac{\beta}{\ln R_m^{\frac{1}{2}}}, \quad \text{where} \quad \beta = \frac{k^2}{2C} = 61{\cdot}349\ldots. \tag{4.7}$$

It is interesting that this expression is independent of $Q$. It is also much less than unity as long as $R_m$ is large, and it may be verified that (4.7) does in fact give the maximum value of $R_1$ as a function of $Q$ for fixed $R_m$. This shows clearly that $Q$ does not have to be small for the analysis to be valid: it is necessary only that $\ln R_m = O(\epsilon^{-2})$. For large fluxes we therefore have a limiting form of the $R$, $\epsilon$ curve described by

$$R = R_0 + 12{\cdot}56\epsilon^2 + \beta/\ln R_m^{\frac{1}{2}} \tag{4.8}$$

and this evidently leads to a minimum $R_\infty$ of $R$ as a function of $\epsilon$ which occurs for $p_3 \gg 1$ when

$$\epsilon = \epsilon_\infty \simeq \left(\frac{\beta}{12{\cdot}56}\right)^{\frac{1}{2}} \frac{1}{\ln p_3}.$$

For this value of $\epsilon$,

$$R = R_\infty \simeq R_0 + 2\beta/\ln p_3. \tag{4.9}$$

Since $R_\infty$ is independent of $Q$, it is much less than $R_c$ in the large-$Q$ limit. So for these strong fields we have a subcritical instability. Thus the form of the instability depends crucially on the size of $Q$. It is easily shown that, for smaller values of $Q$, $R$ has a minimum value $R_{\min}$ as a function of $\epsilon$ that is less than $R_\infty$. Since the bifurcation is supercritical for small enough $Q$ there must be some value $Q_{\text{int}}$ of $Q$ for which

$$R_c(Q) = R_{\min}(Q).$$

$Q_{\text{int}}$ clearly depends on $p_3$ and no simple formula is available. The fact that (3.27) possesses a minimum at all in the high-$R_m$ range means that, for $Q < Q_{\text{int}}$ at least, $R$ begins at $R_c$ and then must reach a local minimum that is greater than $R_c$! Thus the direction of the $R$, $\epsilon$ curve must change at least twice (as sketched in figure 1) or possibly more times if the bifurcation is subcritical. Of course we have no right to expect monotonic behaviour of the curve since we are well into the nonlinear regime.

### 4.4. *Numerical results*

In order to keep the necessary data within reasonable bounds, it was decided to examine only one (large) value of $p_3$. Then for given values of $\epsilon$ and $Q$, (3.21) and (3.27) could be solved to obtain graphs of each of these quantities against $R$. The remaining results, those for $\epsilon$ and the maximum field as functions of $Q$ for fixed $R$, were obtained by interpolation. The value of $p_3$ chosen was $4 \times 10^8$; the range of validity of the theory increases with $p_3$ and a large value of this parameter is not unreasonable in the solar context owing to the large radiative contribution to $\kappa$. For this value we find that

$$R_\infty = R_0 + 7{\cdot}1. \tag{4.10}$$

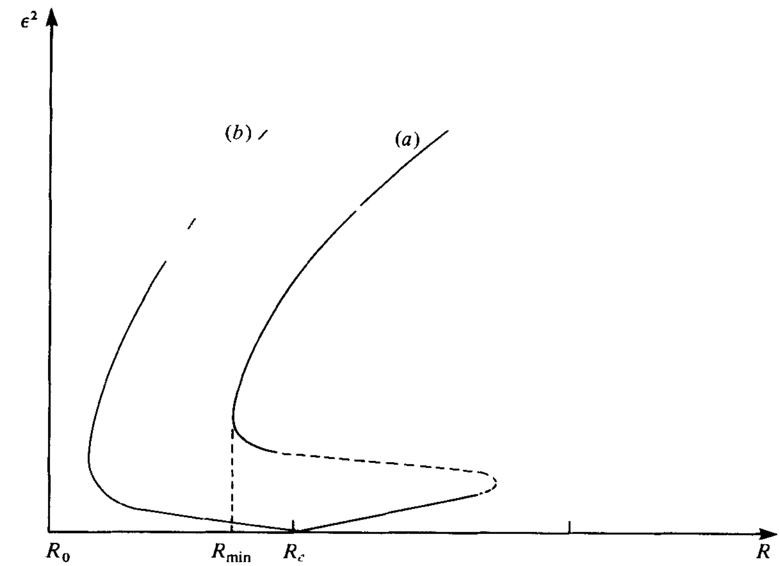


FIGURE 1. Sketch of possible forms of $R$ as a function of $\epsilon^2$ in the axisymmetric geometry. Note that the curve must have form (a) if $R_{\min} > R_c$ (i.e. if $Q < Q_{\mathrm{int}}$). The bifurcation is always supercritical, as in (a), for all sufficiently small $Q$.

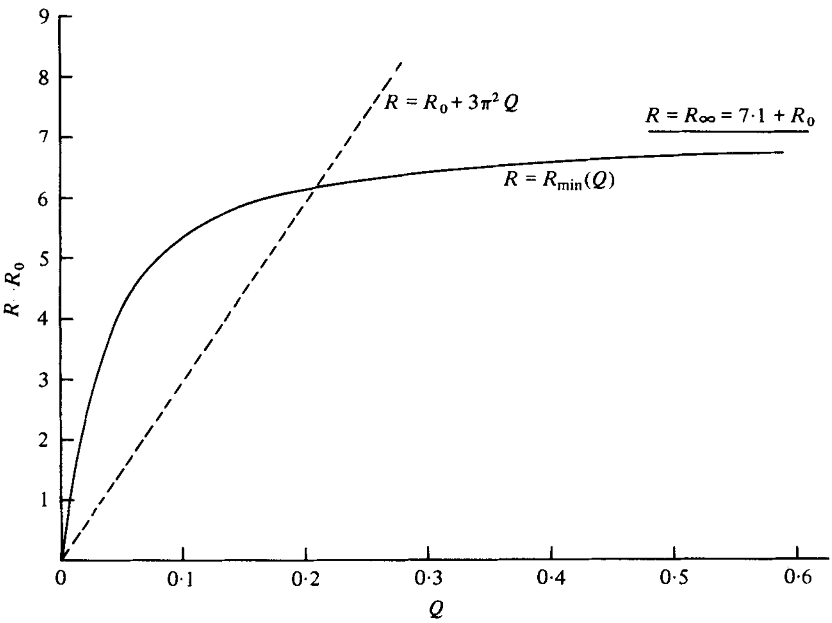


FIGURE 2. $R_{\min}$ and $R_c$ as functions of $Q$ for $p_3 = 4 \times 10^8$.

Figure 2 shows the minimum Rayleigh number $R_{\min}$ for which convection can occur at large $R_m$ plotted as a function of $Q$. As $Q \to \infty$, $R_{\min} \to R_\infty$ from below. The curve also shows $R_c$, the eigenvalue for the exchange of stabilities. The intersection of these lines gives $Q_{\mathrm{int}}$, which for the given value of $p_3$ is 0·2075. In figure 3 the Péclet number $\epsilon$ is plotted as a function of $R$ for $Q = 0{\cdot}9$ and 0 (greater than and less than $Q_{\mathrm{int}}$,

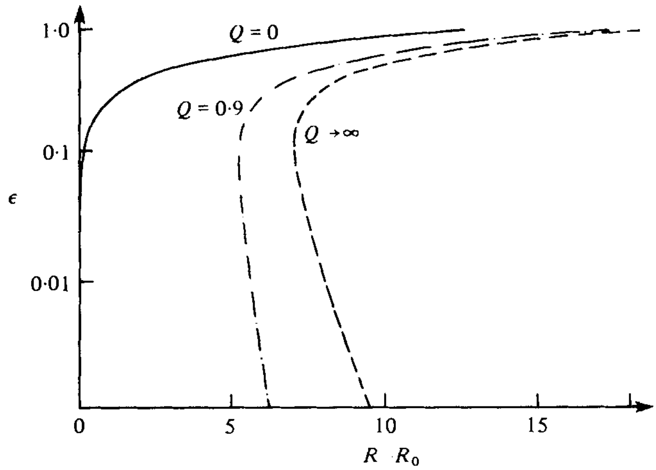


FIGURE 3. $\epsilon$ as a function of $R$ for $Q = 0$, $Q = 0{\cdot}9$ and $Q \to \infty$; $p_3 = 4 \times 10^8$.

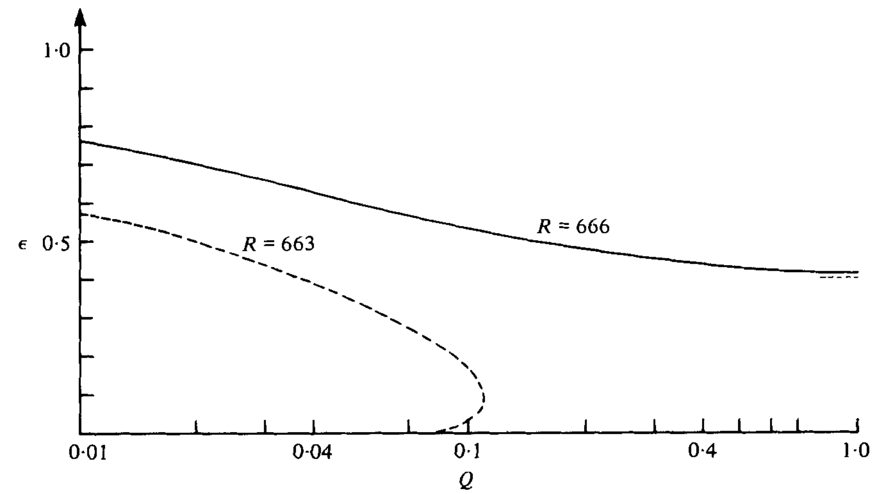


FIGURE 4. $\epsilon$ as a function of $Q$ for $R = 663$ and $R = 666$; $p_3 = 4 \times 10^8$.

respectively) and for $Q \to \infty$, in which case the limiting equation (4.8) applies. The corresponding values of $R_c$ are also given. Figure 4 shows $\epsilon$ as a function of $Q$ for $R = 663$ and 666 (less than and greater than $R_\infty$, respectively). The difference in the two curves is marked. For $R = 663$ there is a maximum value of $Q$ for which convection is possible in the flux rope regime; this gives a curve qualitatively similar to those of Busse. For $R = 666$, on the other hand, convection is possible for all $Q$ (up to the limit $O(R_m/\ln R_m)$ beyond which the theory ceases to apply), and $\epsilon$ asymptotes to the appropriate point on the curve (4.7) ($\epsilon \to 0{\cdot}4$ in this case). The lower part of the $R = 663$ curve corresponds to a negative value of $\partial R/\partial \epsilon$ and is therefore certainly unstable. The corresponding part of the $R = 666$ curve is too near the axis to be shown. Finally, figure 5 shows the graph of $Q^{\frac{1}{2}} R_m p(0)$, a measure of the maximum field, against $Q^{\frac{1}{2}}$, a measure of the mean field, for the same two values of $R$. Clearly, there is in each

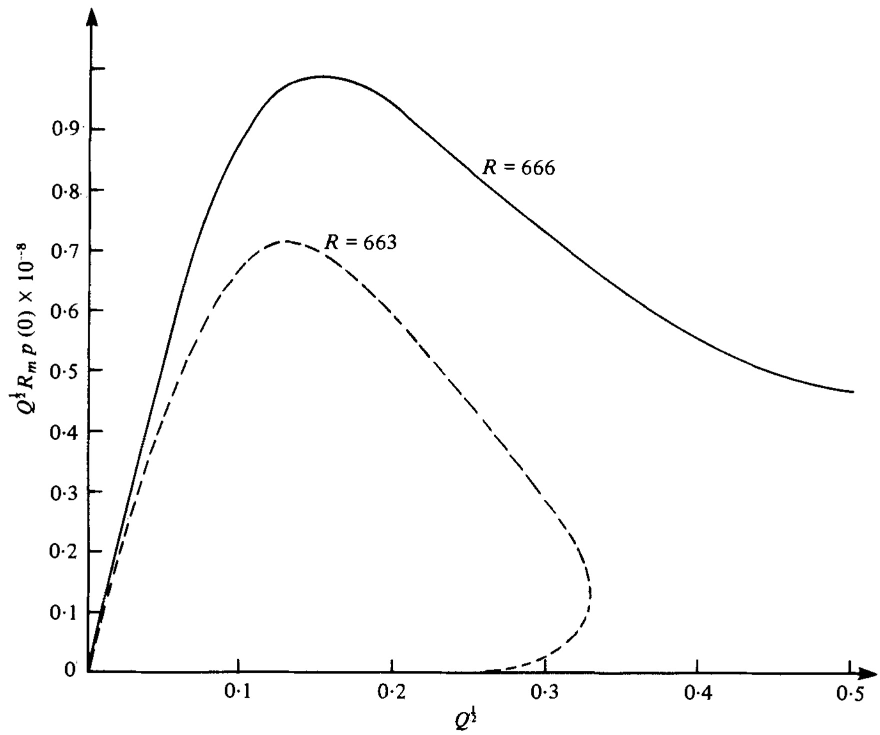


FIGURE 5. $Q^{\frac{1}{2}}R_m p(0)$ (proportional to the peak magnetic field) as a function of $Q^{\frac{1}{2}}$ (proportional to the mean magnetic field) for $R = 663$ and $R = 666$; $p_3 = 4 \times 10^8$.

case a maximum of the field as a function of $Q$. This was to be expected since the same phenomenon occurs in the Oberbeck problem treated by Galloway *et al.* (1978). For $R = 663$, the flux-rope solution 'runs out' at $Q \simeq 0{\cdot}109$, but for the higher value of $R$ the maximum field $\sim Q^{-\frac{1}{4}}$ for large $Q$.

## 5. Conclusions

In previous sections we have shown how the boundary-layer methods developed by Galloway *et al.* (1978) may be applied to the Rayleigh–Bénard problem in a cylindrical geometry. The only significant restrictions on the analysis are that the magnetic Reynolds number remains large and that the Péclet and Reynolds numbers remain small. Because of the singular nature of the axisymmetric flux rope there is no restriction on the amplitude of dynamical effects local to the rope. In the analogous two-dimensional problem studied by Busse, it is only to obtain results possible when the local dynamical effects are small and the velocity is essentially unchanged by the presence of the field. We have shown that there are important qualitative differences between the two geometries. There is for both cases a minimum of $R$ as a function of $\epsilon$ in the flux-rope regime ($R_m \gg 1$). In the axisymmetric case this minimum $R_{\min}$, is greater than $R_c$, the exchange-of-stabilities eigenvalue, for sufficiently small $Q$ but is smaller than $R_c$ for larger values of $Q$. Thus there is no question of subcritical instability for sufficiently weak fields. Furthermore, for $R_{\min} > R_c$ the $R$, $\epsilon$ curve must change direction at least twice as shown in figure 1. Another difference between the geometries is that for large enough $R$ ($R > R_\infty$, see previous section) convection is possible for all values of the flux threading the convective system provided that the flux rope remains thin. The actual assumptions made are that $\eta \ll \kappa \ll \nu$, that $R_m \gg 1$, but that $Pe \ll 1$ and the Reynolds number is vanishingly small. $Q$ may be as large as $p_3/\ln p_3$ for the analysis to be valid.

In this paper we have confined ourselves to the investigation of *steady* solutions of the governing equations. It is well known that unless $Q$ is very small (of order $p_3^{-1}$) there exist overstable oscillations of the system for values $R_{\text{o.s.}}$ of $R$ less than $R_c$. [For large $p_3$ and large $p_1$ we have $R_{\text{o.s.}} = R_0 + O(p_3^{-1}) + O(Qp_3^{-1})$.] On the scale of figure 3, this corresponds to an almost horizontal line passing almost through the origin! Now $R_{\text{o.s.}}$ increases with $Q$ and one would be tempted to conclude that, for sufficiently large $Q$, $R_{\text{o.s.}} > R_{\min}$, which would strongly suggest that steady finite amplitude instability was possible. Unfortunately, for large $Q$, $R_{\min} \simeq R_\infty = R_0 + O(1/\ln p_3)$. Thus $R_{\text{o.s.}} \simeq R_\infty$ when $Q \sim p_3/\ln p_3$, i.e. precisely at the point where the flux is so large that the flux-rope approximation breaks down for $\epsilon \simeq 1$. So no information can be obtained in this way. The nonlinear behaviour of the overstable solution has not been investigated analytically, but the discovery by Galloway (1977) and Galloway & Moore (1978) that for large Péclet numbers steady flux-rope regimes are indeed realized when one might expect overstability indicates that $R_{\text{o.s.}}$ increases with $\epsilon$.

We cannot, therefore, claim to have solved the problem of subcritical instability in magnetoconvection. We have, however, made significant analytic progress towards an understanding of the steady solution. We feel that the methods employed will prove useful in understanding the nonlinear interaction of magnetic fields and compressible convection (which is, of course, much more relevant in the context of the solar photosphere, whose understanding forms the motivation for the present work). In the compressible problem there is the added effect of magnetic pressure, which will reduce the density in the flux rope, so the outcome is by no means obvious. It would also be most desirable to extend the theory to a hexagonal convection pattern so as to model the convection zone more closely, but this appears to require significant computational effort.

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